§4.8 坐标与基变换

Def 1. 设 V 是数域 P 上的 n 维线性空间，

$\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n$ 是 V 的一组基，

$\vec{\beta} \in V$，有唯一的数组 $x_1, x_2, \cdots, x_n$，

s.t. $\vec{\beta} = \sum_{i=1}^{n} x_i \vec{\alpha}_i$. 我们称 $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in P^{n\times 1}$

为 $\vec{\beta}$ 在基 $\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n$ 下的<u>坐标</u>. 记作

$$\mathrm{crd}(\vec{\beta}; \vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

于是立刻得到

1° $\vec{\beta}_1 = \vec{\beta}_2 \Leftrightarrow \mathrm{crd}(\vec{\beta}_1, \cdot) = \mathrm{crd}(\vec{\beta}_2, \cdot)$.

2° $\forall \vec{X} \in \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\exists!\ \vec{\beta} \in V$, s.t. $\mathrm{crd}\vec{\beta} = \vec{X}$.

3° $\mathrm{crd}(\vec{\beta}_1 + \vec{\beta}_2, \cdot) = \mathrm{crd}(\vec{\beta}_1, \cdot) + \mathrm{crd}(\vec{\beta}_2, \cdot)$, $\forall \vec{\beta}_1, \vec{\beta}_2 \in V$ …保持加法

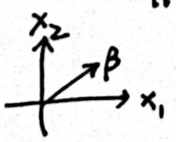
4° $\mathrm{crd}(k\vec{\beta}) = k\,\mathrm{crd}(\vec{\beta})$, $\forall k \in P$, $\vec{\beta} \in V$. …保持数乘

5° V 中向量组 $\beta_1 \cdots \beta_k$ 线性相关 $\Leftrightarrow$ $\mathrm{crd}\beta_1, \mathrm{crd}\beta_2, \cdots, \mathrm{crd}\beta_k$ 在 $P^{n\times 1}$ 中线性相关.

于是可以由 $P^{n\times 1}$ 完全描绘 V 的性质.

技巧：展开"线性组合"：取 V 的基 $\vec{\alpha}_1 \cdots \vec{\alpha}_n$，$\vec{\beta} \in V$. 设

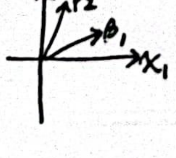
倘若 $\beta = \sum_{i=1}^{n} x_i \vec{\alpha}_i$，则 $\mathrm{crd}\,\beta = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. 记



$$\beta = (\vec{\alpha}_1 \cdots \vec{\alpha}_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (\vec{\alpha}_1 \cdots \vec{\alpha}_n)\ \mathrm{crd}(\vec{\beta}; \vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n).$$

$$\overset{\text{简记}}{=} (\vec{\alpha}_1, \cdots, \vec{\alpha}_n)\ \mathrm{crd}\,\vec{\beta}.$$

如果有一组 $\beta_1 \cdots \beta_k \in V$，且 $\mathrm{crd}\,\beta_j = \vec{X}_j$，可记



$$(\beta_1, \beta_2, \cdots, \beta_k) = (\alpha_1, \alpha_2, \cdots, \alpha_n)(\vec{X}_1, \cdots, \vec{X}_k).$$

此时，如果 $\beta = \sum_{j=1}^{k} b_j \beta_j$ ，那么 $\mathrm{Crd}\,\beta = \mathrm{crd} \sum_{j=1}^{k} b_j \vec{\beta_j}$

$\cancel{\mathrm{crd}}$

$$= \sum_{j=1}^{k} b_j \overset{\vec{X_j}}{\boxed{\mathrm{crd}\,\beta_j}}$$

$$= (\vec{X_1}, \vec{X_2}, \cdots, \vec{X_k}) \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix}$$

从而

$$\vec{\beta} = (\beta_1, \beta_2, \cdots, \beta_k) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix}$$

$$= (\alpha_1, \alpha_2, \cdots, \alpha_n) \boxed{(X_1, X_2, \cdots, X_k)} \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix}. \qquad (*)$$

Def 2. 设 $\alpha_1 \cdots, \alpha_n$ 与 $\beta_1, \cdots, \beta_n$ 是 $\mathbb{P}$ 上 n 维线性空间 V 的两组基，设 β_j 在 $\vec{\alpha_1}, \vec{\alpha_2}, \cdots, \alpha_n$ 下的坐标为 T_j . 即

$$\mathrm{crd}(\beta_j; \alpha_1, \alpha_2, \cdots, \alpha_n) = T_j \quad , 1 \le j \le n.$$

称矩阵 $(T_1, T_2, \cdots, T_n)$ 为从基 $\vec{\alpha_1}, \vec{\alpha_2}, \cdots, \vec{\alpha_n}$ 到 $\vec{\beta_1}, \vec{\beta_2}, \cdots, \vec{\beta_n}$ 的<u>过渡矩阵</u>. 记为 $T\begin{pmatrix} \vec{\alpha_1}, \cdots, \vec{\alpha_n} \\ \vec{\beta_1}, \cdots, \vec{\beta_n} \end{pmatrix}$.

Th 1. 设 $\alpha_1 \cdots \alpha_n$ 与 $\beta_1 \cdots \beta_n$ 均为 V 的基，又 $\vec{\gamma} \in V$，则

$$\mathrm{crd}(\vec{\gamma}; \vec{\alpha_1}, \cdots, \vec{\alpha_n}) = T\begin{pmatrix} \vec{\alpha_1} & \cdots & \vec{\alpha_n} \\ \vec{\beta_1} & \cdots & \vec{\beta_n} \end{pmatrix} \mathrm{crd}(\vec{\gamma}; \vec{\beta_1}, \cdots, \vec{\beta_n}).$$

Proof: $\gamma = (\vec{\alpha_1}, \vec{\alpha_2}, \cdots, \vec{\alpha_n})\, \mathrm{crd}(\vec{\gamma}; \vec{\alpha_1}, \cdots, \vec{\alpha_n})$

$$= \underbrace{(\vec{\beta_1}, \vec{\beta_2}, \cdots, \vec{\beta_n})}\, \mathrm{crd}(\vec{\gamma}; \vec{\beta_1}, \cdots, \vec{\beta_n})$$

重写(*): $\vec{\beta} = (\alpha_1, \cdots, \alpha_n) \underbrace{(X_1, \cdots, X_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}}_{T\begin{pmatrix} \vec{\alpha_1} \cdots \vec{\alpha_n} \\ \vec{\beta_1} \cdots \vec{\beta_n} \end{pmatrix}}$

$$= \underbrace{(\vec{\alpha_1}, \vec{\alpha_2}, \cdots, \vec{\alpha_n})\, T\begin{pmatrix} \vec{\alpha_1}, \cdots, \vec{\alpha_n} \\ \vec{\beta_1}, \cdots, \vec{\beta_n} \end{pmatrix}}\ \mathrm{crd}(\vec{\gamma}, \vec{\beta_1}, \cdots, \vec{\beta_n}).$$

Th2. 设 $\vec{\alpha_1},\cdots,\vec{\alpha_n}$；$\vec{\beta_1}\cdots\vec{\beta_n}$，$\vec{\gamma_1},\cdots,\vec{\gamma_n}$ 为V的三组基，则

$$T\begin{pmatrix}\vec{\alpha_1}\cdots\vec{\alpha_n}\\ \vec{\gamma_1}\cdots\vec{\gamma_n}\end{pmatrix} = T\begin{pmatrix}\vec{\alpha_1}\cdots\vec{\alpha_n}\\ \vec{\beta_1}\cdots\vec{\beta_n}\end{pmatrix} T\begin{pmatrix}\vec{\beta_1}\cdots\vec{\beta_n}\\ \vec{\gamma_1}\cdots\vec{\gamma_n}\end{pmatrix}$$

Proof. 由 Th 1 得：

$\forall j$，$\quad crd(\vec{\gamma_j};\vec{\alpha_1},\cdots,\vec{\alpha_n}) = T\begin{pmatrix}\vec{\alpha_1}\cdots\vec{\alpha_n}\\ \vec{\beta_1}\cdots\vec{\beta_n}\end{pmatrix} crd(\vec{\gamma_j};\vec{\beta_1},\vec{\beta_2},\cdots,\vec{\beta_n})$.

于是两端矩阵每一列都相同.

推论：$T\begin{pmatrix}\alpha_1\cdots\alpha_n\\ \beta_1\cdots\beta_n\end{pmatrix}^{-1} = T\begin{pmatrix}\beta_1\cdots\beta_n\\ \alpha_1\cdots\alpha_n\end{pmatrix}$.

Eg. 求 $\vec{\alpha}=(a_1,a_2,\cdots,a_n)\in P^{1\times n}$ 在 $\begin{aligned}\vec{\alpha_1}&=(1,1,\cdots,1)\\ \vec{\alpha_2}&=(0,1,\cdots,1)\\ &\vdots\\ \vec{\alpha_n}&=(0,0,\cdots,1)\end{aligned}$ 下的坐标.

解. $T\begin{pmatrix}\alpha_1,\alpha_2,\cdots,\alpha_n\\ \varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n\end{pmatrix} = T\begin{pmatrix}\varepsilon_1,\cdots\varepsilon_n\\ \alpha_1,\cdots\alpha_n\end{pmatrix}^{-1} = \begin{pmatrix}1&0&\cdots&0\\ 1&1&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ 1&1&\cdots&1\end{pmatrix}^{-1}$

$$crd(\vec{\alpha};\alpha_1,\cdots,\alpha_n) = T\begin{pmatrix}\alpha_1\cdots\alpha_n\\ \varepsilon_1\cdots\varepsilon_n\end{pmatrix}\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}$$

$$= \begin{pmatrix}1&0&\cdots&0\\ 1&1&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ 1&1&\cdots&1\end{pmatrix}^{-1}\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix} = \begin{pmatrix}a_1\\ a_2-a_1\\ \vdots\\ a_n-a_{n-1}\end{pmatrix}$$

~~Eg:~~