§ 5.3 线性变换的矩阵.

Th 1. 设 $\alpha_1 \cdots \alpha_n$ 是 $\mathbb{P}$ 上 n 维线性空间的一组基，则对于 V 中任意 n 个向量 $\beta_1 \cdots \beta_n$ 存在唯一的线性变换 $\mathscr{A}$, s.t.

$$\mathscr{A}\alpha_i = \beta_i \quad , 1 \le i \le n.$$

Prf. 我们希望通过"每个 β_i 都可唯一地被 $\alpha_1 \cdots \alpha_n$ 线性组合"一事说明之. 干脆先取 $\mathscr{A}\left(\sum_{i=1}^{n} k_i \alpha_i\right) = \sum_{i=1}^{n} k_i \beta_i$，然后论证其性质.

对于 $\alpha = \sum_{i=1}^{n} k_i \alpha_i$, $\beta = \sum_{i=1}^{n} l_i \alpha_i \in V$, $k, l \in \mathbb{P}$, 有

$$\mathscr{A}(k\alpha + l\beta) \xlongequal{\text{代}\alpha,\beta} \mathscr{A}\left(\sum_{i=1}^{n} (kk_i + ll_i)\alpha_i\right)$$

$$\xlongequal{\mathscr{A}\text{作用}} \sum_{i=1}^{n} (kk_i + ll_i)\beta_i$$

$$\xlongequal{\text{拆开}\Sigma} k\boxed{\sum_{i=1}^{n} k_i\beta_i} + l\underbrace{\sum_{i=1}^{n} l_i\beta_i} = k\boxed{\mathscr{A}\alpha} + l\underbrace{\mathscr{A}\beta}$$

故 $\mathscr{A} \in \mathrm{End}\,V$, 假使还有一变换 $\mathscr{B} \in \mathrm{End}\,V$, 且 $\mathscr{B}\alpha_i = \beta_i$, $1 \le i \le n$,

则 $\mathscr{B}\left(\sum_{i=1}^{n} k_i \alpha_i\right) = \sum_{i=1}^{n} k_i \mathscr{B}\alpha_i = \sum_{i=1}^{n} k_i \beta_i = \mathscr{A}\left(\sum_{i=1}^{n} k_i \alpha_i\right)$.

此说明 $\mathscr{A} = \mathscr{B}$，此变换是唯一的.

如若 $\alpha_i = \beta_i$ 时, $\mathscr{A} = \mathrm{id}$, $\beta_i = 0$ 时, $\mathscr{A} = 0$.

Def 1. 设 $\alpha_1 \cdots \alpha_n$ 是 $\mathbb{P}$ 上 n 维线性空间的一组基，记

$$\mathrm{crd}(\alpha; \alpha_1 \cdots \alpha_n) = \mathrm{crd}\,\alpha \quad , \alpha \in V.$$

又 $\mathscr{A} \in \mathrm{End}\,V$，称矩阵

$(\mathrm{crd}\,\mathscr{A}\alpha_1 , \mathrm{crd}\,\mathscr{A}\alpha_2 , \cdots , \mathrm{crd}\,\mathscr{A}\alpha_n)$ 为 $\mathscr{A}$ 在基 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 下的矩阵. 记为

$$M(\mathscr{A}; \alpha_1 , \alpha_2 , \cdots , \alpha_n).$$

可简记为 $M(\mathscr{A})$.

也就是说，若

$$M(\mathscr{A};\alpha_1,\alpha_2,\cdots,\alpha_n)=\begin{pmatrix}a_{11}&\cdots&a_{1n}\\ \vdots&\ddots&\vdots\\ a_{n1}&\cdots&a_{nn}\end{pmatrix},\ \forall 1\le j\le n \text{ 有}$$

$$\begin{aligned}\mathscr{A}\alpha_j &= a_{1j}\alpha_1+a_{2j}\alpha_2+\cdots+a_{nj}\alpha_n\\ &=(\alpha_1,\alpha_2,\cdots,\alpha_n)\ \mathrm{col}_j\, M(\mathscr{A}).\end{aligned}$$

Th2、设 $\alpha_1,\alpha_2,\cdots,\alpha_n$ 是 $\mathbb{P}$上线性空间V的一组基，$\mathscr{A}\in \mathrm{End}V$，

则 $\mathrm{crd}(\mathscr{A}\alpha;\alpha_1,\cdots,\alpha_n)=M(\mathscr{A};\alpha_1\cdots\alpha_n)\,\mathrm{crd}(\alpha;\alpha_1\cdots\alpha_n)$.

Prf. $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)\,\mathrm{crd}\,\alpha$

$$\begin{aligned}\mathscr{A}\alpha &= (\mathscr{A}\alpha_1,\mathscr{A}\alpha_2,\cdots,\mathscr{A}\alpha_n)\,\mathrm{crd}\,\alpha\\ &=(\alpha_1,\alpha_2,\cdots,\alpha_n)\underbrace{(\mathrm{crd}\,\mathscr{A}\alpha_1,\mathrm{crd}\,\mathscr{A}\alpha_2,\cdots,\mathrm{crd}\,\mathscr{A}\alpha_n)}_{M(\mathscr{A})}\,\mathrm{crd}\,\alpha\end{aligned}$$

因而 $\mathrm{crd}\,\mathscr{A}\alpha = M(\mathscr{A})\,\mathrm{crd}\,\alpha$.

Th3.（EndV与$\mathbb{P}^{n\times n}$的关系）设 V是$\mathbb{P}$上 n维线性空间，

$\alpha_1\cdots\alpha_n$ 为V的一组基，则 $\mathrm{End}V\to\mathbb{P}^{n\times n}$ 的映射

$$\varphi(\mathscr{A})=M(\mathscr{A};\alpha_1\cdots\alpha_n),\ \forall\mathscr{A}\in\mathrm{End}V.$$

满足如下条件.

1) φ是 $\mathrm{End}V\to\mathbb{P}^{n\times n}$ 的同构映射

2) $\varphi(\mathscr{A}\mathscr{B})=\varphi(\mathscr{A})\varphi(\mathscr{B})$

3) $\mathscr{A}\in GL(V)\iff\varphi(\mathscr{A})$可逆，且 $\varphi(\mathscr{A}^{-1})=\varphi(\mathscr{A})^{-1}$.

4) $\forall f(x)\in\mathbb{P}[x]$，$\varphi(f(\mathscr{A}))=f(\varphi(\mathscr{A}))$. $\mathscr{A}\in\mathrm{End}V$.

Prf. ① 由Th1，$\mathscr{A}=\mathscr{B}\iff\mathscr{A}\alpha_i=\mathscr{B}\alpha_i\iff\mathrm{crd}\,\mathscr{A}\alpha_i=\mathrm{crd}\,\mathscr{B}\alpha_i\iff M(\mathscr{A})=M(\mathscr{B})$

即 $\varphi(\mathscr{A})=\varphi(\mathscr{B})$. 因此$\varphi$是一一的（单射）

又若 $A\in\mathbb{P}^{n\times n}$，有 $\beta_i\in V$, s.t. $\mathrm{crd}\,\beta_i=\mathrm{col}_i A$. 设 $\mathscr{A}\in\mathrm{End}V$，使 $\mathscr{A}\alpha_i=\beta_i$，于是 $\varphi(\mathscr{A})=M(\mathscr{A})=A$. 故$\varphi$满，即~~一一的~~双射.

设 $\mathscr{A},\mathscr{B}\in\mathrm{End}V$，$k,l\in\mathbb{P}$，由

$$\mathrm{crd}((k\mathscr{A}+l\mathscr{B})\alpha_i)=\mathrm{crd}(k\mathscr{A}\alpha_i+l\mathscr{B}\alpha_i)=k\,\mathrm{crd}\,\mathscr{A}\alpha_i+l\,\mathrm{crd}\,\mathscr{B}\alpha_i$$

知 $\varphi(k\mathscr{A}+l\mathscr{B})=k\varphi\mathscr{A}+l\varphi\mathscr{B}$. 是线性空间的同构.

② 若 $\mathscr{A},\mathscr{B}\in \mathrm{End}\,V$，由 Th2，

$$\mathrm{crd}\,\mathscr{A}\mathscr{B}\alpha_i = \mathrm{crd}\,\mathscr{A}(\mathscr{B}\alpha_i) = M(\mathscr{A})\,\mathrm{crd}\,\mathscr{B}\alpha_i$$
$$= M(\mathscr{A})\,\mathrm{col}_i M(\mathscr{B})$$

从而 $M(\mathscr{A}\mathscr{B}) = M(\mathscr{A})M(\mathscr{B})$.

③ $\mathscr{A}\in GL(V)$，$\mathscr{A}\mathscr{A}^{-1}=id$，则 $\varphi(\mathscr{A})\varphi(\mathscr{A}^{-1}) = \varphi(id) = I_n$

从而 $\varphi(\mathscr{A}^{-1}) = \varphi(\mathscr{A})^{-1}$. …

④ 由1°与2°及定义可知.

Coll1. 若 $\dim V = n$，则 $\dim(\mathrm{End}\,V) = n^2$. 因为 $\mathrm{End}\,V$ 与 $\mathbb{P}^{n\times n}$ 同构.

Coll2. 设 $\dim V = n$，$\mathscr{A}\in \mathrm{End}\,V$，则 $d_{\mathscr{A}}(x)$ 存在.

◁ 由 $\mathscr{A}^0 = id$，$\mathscr{A}^1 = \mathscr{A}$，…，$\mathscr{A}^{n^2}$ 为 $\mathrm{End}\,V$ 中 n^2+1 个元素. 因而线性相关. 有不全为0之数，$\sum_{i=0}^{n^2} a_i\mathscr{A}^i = 0$. 因而 $f(x) = \sum_{i=0}^{n^2} a_i x^i \in \mathbb{P}[x]$. 且 $f(x)\neq 0$，$f(\mathscr{A})=0$，故 $\ker\varphi_{\mathscr{A}} \neq \{0\}$ ▷

这与§5.2 推论2 呼应得很好.

Th 4. (线性变换的基变换) 设 $\alpha_1\cdots\alpha_n$，$\beta_1\cdots\beta_n$ 是 $\mathbb{P}$ 上 $\boxed{\text{v.s.}}$ V 的基，
，线性空间 vector space

$T = T\begin{pmatrix}\alpha_1 \cdots \alpha_n \\ \beta_1 \cdots \beta_n\end{pmatrix}$ 是从 $\alpha_1\cdots\alpha_n$，$\beta_1\cdots\beta_n$ 的过渡矩阵.

倘若有 $\mathscr{A}\in \mathrm{End}\,V$，则 $M(\mathscr{A};\beta_1,\beta_2,\cdots,\beta_n) = T^{-1}M(\mathscr{A};\alpha_1\cdots\alpha_n)\,T$.

另一方面，若 $C\in\mathbb{P}^{n\times n}$，且存在可逆矩阵 U，使 $U^{-1}CU = M(\mathscr{A};\alpha_1\cdots\alpha_n)$，则在 V 中有基 $\gamma_1\cdots\gamma_n$ 使得 $M(\mathscr{A};\gamma_1\cdots\gamma_n) = C$.

Proof. 为简单记，$A = M(\mathscr{A};\alpha_1\cdots\alpha_n)$，$B = M(\mathscr{A};\beta_1,\cdots,\beta_n)$.

$$\mathrm{col}_j B = \mathrm{crd}(\mathscr{A}\beta_j;\beta_1,\beta_2,\cdots,\beta_n) = \underset{\boxed{\text{将}\beta\text{基转为}\alpha}}{T^{-1}}\,\mathrm{crd}(\mathscr{A}\beta_j;\alpha_1\cdots\alpha_n)$$

$$\xlongequal{\text{Th2.展开}} T^{-1}\,\mathrm{crd}(\mathscr{A};\alpha_1\cdots\alpha_n)\,\mathrm{crd}(\beta_j;\alpha_1\cdots\alpha_n)$$
$$= T^{-1}A\,\mathrm{col}_j T$$

于是合一起即为 $B = T^{-1}AT$.

设 $S\in\mathbb{P}^{n\times n}$，$S$可逆，$S^{-1}AS=C$，令 $\gamma_j=(\alpha_1,\alpha_2,\cdots,\alpha_n)\,\mathrm{col}_j S$，

由S可逆，知$\gamma_1\cdots\gamma_n$为V的基，且

$$S=T\begin{pmatrix}\alpha_1\cdots\alpha_n\\ \gamma_1\cdots\gamma_n\end{pmatrix}，从而\ M(\mathscr{A};\gamma_1\cdots\gamma_n)=S^{-1}AS=C.$$

Def2、设 $\mathscr{A}$，$\mathscr{B}\in\mathbb{P}^{n\times n}$，$\exists T\in\mathbb{P}^{n\times n}$ s.t. $T^{-1}AT=B$.

则称A与B相似．记为$A\sim B$．

性质
- $A\sim A$
- $A\sim B\Rightarrow B\sim A$
- $A\sim B$，$B\sim C\Rightarrow A\sim C$

$\Big\}$ 等价关系

- $A\sim B$，$f(x)\in\mathbb{P}[x]$，则 $f(\mathscr{A})=f(B)$
- $A\sim B$，$\Rightarrow \det A=\det B$

  $\operatorname{tr}A=\operatorname{tr}B$．